Implicit Surfaces

Implicit surfaces

- Explicit functions

  $y = f(x)$ e.g.: $y = x^2$
- Parametric function

  $\mathbf{Q}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ e.g.: $\mathbf{Q}(t) = \begin{pmatrix} t^3+1 \\ t^2-4t \end{pmatrix}$
- Implicit function

  $f(x,y) = 0$ e.g.: $x^2+y^2-r^2=0$

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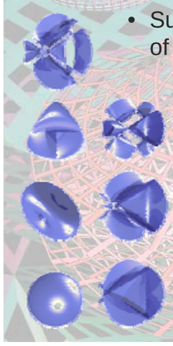
Implicit Surfaces



- Surfaces can be represented as the set of roots (solutions) of

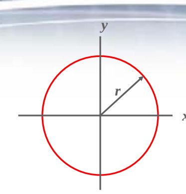
$$f(x,y,z)=0$$

  - In graphics, mostly used for spheres and quadratic surfaces
  - In general, very hard to find a function for an arbitrary shape. parameters often non-intuitive
  - Difficult to join
  - Difficult to render
  - Smooth
  - Easy intersections

Spheres

$$x^2+y^2=r^2$$

$$x^2+y^2-r^2=0$$



$$x^2+y^2+z^2=r^2$$

$$|\mathbf{X}-\mathbf{C}|^2=r^2$$

Generalized sphere around C with radius r

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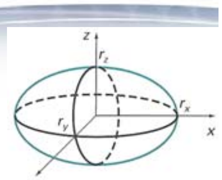
Ellipsoid

$$\left(\frac{x}{r_x}\right)^2+\left(\frac{y}{r_y}\right)^2+\left(\frac{z}{r_z}\right)^2=1$$



- Generalized ellipsoid:

$$(\mathbf{X}-\mathbf{C})^{\mathbf{T}}\mathbf{R}^{\mathbf{T}}\mathbf{D}\mathbf{R}(\mathbf{X}-\mathbf{C})=1$$

D diagonal matrix with entries $1/r_i^2$ (r_i radius)
R rotation matrix
C center point

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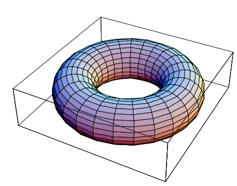
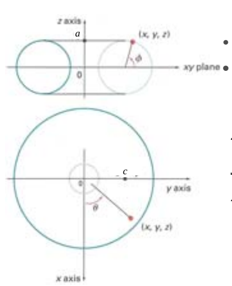
Torus

$$\left(\frac{c}{a}-\sqrt{\left(\frac{x}{a}\right)^2+\left(\frac{y}{a}\right)^2}\right)^2+\left(\frac{z}{a}\right)^2=1$$





- a is tube radius, c 'large' radius
- Exercise: verify the implicit form from the parametric form:

$$\begin{aligned} x &= a(\tfrac{c}{a}+\cos\varphi)\cos\theta & -\pi \le \varphi \le \pi \\ y &= a(\tfrac{c}{a}+\cos\varphi)\sin\theta & -\pi \le \theta \le \pi \\ z &= a\sin\varphi \end{aligned}$$

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Solution

$$\left(\frac{c}{a}-\sqrt{\left(\frac{x}{a}\right)^2+\left(\frac{y}{a}\right)^2}\right)^2+\left(\frac{z}{a}\right)^2=1$$

$$\begin{array}{lcll} x & = & a(\frac{c}{a}+\cos\varphi)\cos\theta & -\pi \le \varphi \le \pi \\ y & = & a(\frac{c}{a}+\cos\varphi)\sin\theta & -\pi \le \theta \le \pi \\ z & = & a\sin\varphi & \end{array}$$

$$\left(\frac{y}{a}\right)^2 = (\tfrac{c}{a}+\cos\varphi)^2\sin^2\theta$$

$$\left(\frac{x}{a}\right)^2 = (\tfrac{c}{a}+\cos\varphi)^2\cos^2\theta$$

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{a}\right)^2 = (\tfrac{c}{a}+\cos\varphi)^2(\cos^2\theta+\sin^2\theta) = (\tfrac{c}{a}+\cos\varphi)^2$$

$$\left(\frac{c}{a}-\sqrt{\left(\frac{x}{a}\right)^2+\left(\frac{y}{a}\right)^2}\right)^2 = \cos^2\varphi$$

$$\left(\frac{z}{a}\right)^2 = \sin^2\varphi$$

$$\left(\frac{c}{a}-\sqrt{\left(\frac{x}{a}\right)^2+\left(\frac{y}{a}\right)^2}\right)^2+\left(\frac{z}{a}\right)^2=1$$

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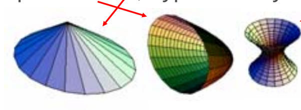
Quadrics

- Quadratic surfaces –or *quadrics*- have the general form

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2px+2qy+2rz+d=0$$

- Based on the parameters a,b,c,e,f,g,p,q,r we can define a large number of classes, e.g.: Ellipsoids, ellipctic cones, elliptic cylinders, elliptic paraboloids, hyperbolic cylinders, hyperboloids



$$\frac{x^2}{a^2+\theta}+\frac{y^2}{b^2+\theta}+\frac{z^2}{c^2+\theta}=1$$

- This form is sometimes also called a quadric:

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Normal vector

- The normal vector of a point on an implicit surface defined by $f(x,y,z)=0$ is given by the gradient vector:

$$\nabla f = \left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right)$$

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Exercise

$$\nabla f = \left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right)$$

- Compute the normal vector in the points

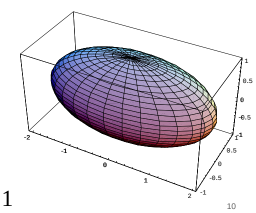
$(2,0,0)$

$(0,0,1)$

$(1,\tfrac{1}{2},\tfrac{1}{2}\sqrt{2})$

of the ellipsoid

$$\left(\frac{x}{2}\right)^2+y^2+z^2=1$$



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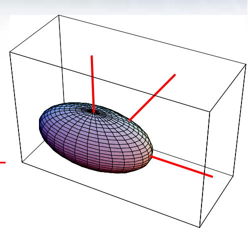
Solution

$$\nabla f = (\tfrac{1}{2}x, 2y, 2z)$$

f	∇f
$(2,0,0)$	$(1,0,0)$
$(0,0,1)$	$(0,0,2)$
$(1,\tfrac{1}{2},\tfrac{1}{2}\sqrt{2})$	$(\tfrac{1}{2},1,\sqrt{2})$



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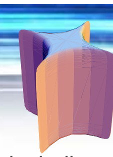
Superquadrics



- Quadrics can be extended by including an exponential parameter
- The most common extensions are the *superellipses* and the *superellipsoids*

$$\left(\left(\frac{x}{r_x}\right)^{2/s_2}+\left(\frac{y}{r_y}\right)^{2/s_2}\right)^{s_2/s_1}+\left(\frac{z}{r_z}\right)^{2/s_1}=1$$

$$\left(\frac{x}{r_x}\right)^{2/s}+\left(\frac{y}{r_y}\right)^{2/s}=1$$

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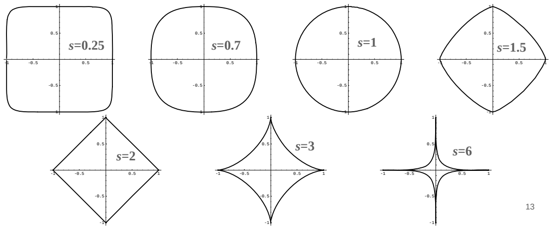
Superellipses

$$\left(\frac{x}{r_x}\right)^{2/s} + \left(\frac{y}{r_y}\right)^{2/s} = 1$$

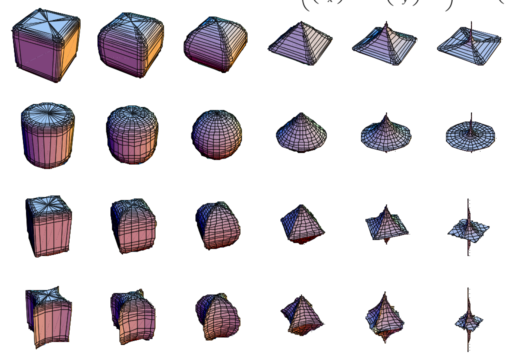
- The parameter s can be interpreted as 'squareness'
  - For $s \to 0$ we have a square
  - $s=1$ give the standard ellipse
  - $s=2$ gives a diamond
  - $s>2$ gives stars

Examples with $r_x=r_y=1$



Superellipsoids

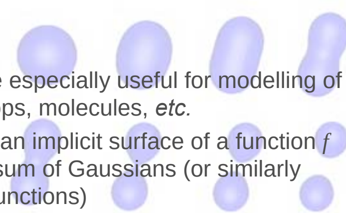
$$\left(\left(\frac{x}{r_x}\right)^{2/s_2} + \left(\frac{y}{r_y}\right)^{2/s_2}\right)^{s_2/s_1} + \left(\frac{z}{r_z}\right)^{2/s_1} = 1$$



Blobs (Metaballs)

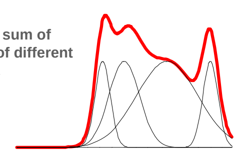
- Implicit surfaces are generally hard to combine into realistic shapes
- An exception is the use of *blobs* which can be combined into 'organic' looking objects

Blobs

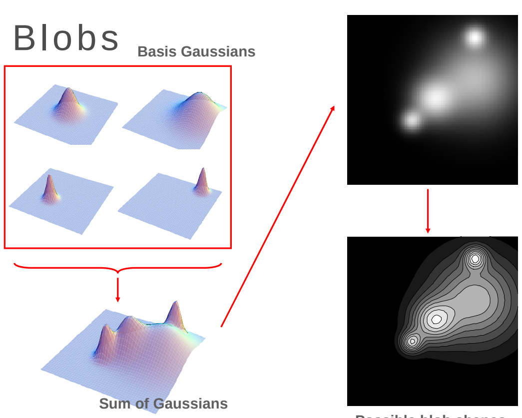


- Blobs are especially useful for modelling of water drops, molecules, *etc*.
- A blob is an implicit surface of a function f that is a sum of Gaussians (or similarly shaped functions)

Red curve is the sum of four Gaussians of different mean and sigma



Blobs

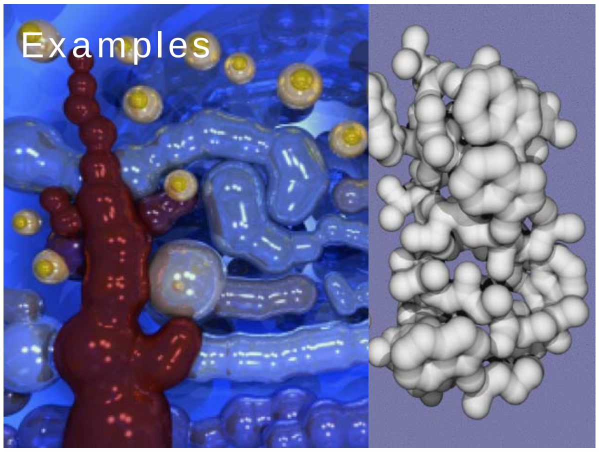


Blobs

- General formula:

$$f(x,y,z) = \sum_i c_i e^{-b_i\left((x-p_i)^2+(y-q_i)^2+(z-r_i)^2\right)} - t = 0$$

- For each Gaussian:
  - c_i determines the height
  - b_i determines the width
  - p_i, q_i, r_i determine the offset (location)
- t determines which level set is the blob surface



Comparison

Feature	Polygonal Mesh	Implicit Surface	Parametric Surface	Subdivision Surface
Accurate	No	Yes	Yes	Yes
Concise	No	Yes	Yes	Yes
Intuitive specification	No	No	Yes	No
Affine invariant	Yes	Yes	Yes	Yes
Arbitrary topology	Yes	No	No	Yes
Guaranteed continuity	No	Yes	Yes	Yes
Natural parameterization	No	No	Yes	No
Efficient intersections	No	Yes	No	No